Fourier's theorem

Ques:- State and Explain Fourier's Theorem what is its application to rectangular wave and show tooth wave?

Ans:→ J.B.J Fourier was the first man to give a satisfactory solution to the problem based on the vibration of strings under tension. In 19th century this series came into time light and since then it has been applied in almost every branch of mathematical Physics. The most important application is in the field of mechanical vibrations.

Fourier's Theorem:→ Fourier's theorem provides a mathematical base to analyse a comlex wave. This is extended as

"Fourier's theorem states that a finite periodic function may be expressed as the sum of different simple harmonic terms, some of them are cosine terms and some of them are sine terms".

In the case of non-periodic function the sum becomes an integral.

The condition imposed are that

(i) The displacement must be single valued and continuous.

(ii) The displacement must be finite.

Since we are dealing with sound waves, these condition are fully satisfied because no particle can have more than one displacement at a time and the displacement cannot be infinite at any stage due to the damping and elastic forces of the medium.

The theorem can be mathematically expressed in the form.

$$y = f(\omega t) = a_0 + a_1 \sin(\omega t + \alpha_1) + a_2 \sin(2\omega t + \alpha_2) + \cdots + a_r \sin(r\omega t + \alpha_r)$$

where,

y is the displacement of a complex periodic vibration of frequency $\frac{\omega}{2\pi}$ and a_0, a_1 ---- are the amplitudes of component simple harmonic vibration and α_1, α_2 --- are the phases.

The theorem can be better expressed in terms of separate sine and cosine terms as follows:-

$$y = A_0 + A_1 \sin\omega t + A_2 \sin 2\omega t + \cdots + A_r \sin r\omega t + B_1 \cos\omega t + B_2 \cos 2\omega t + \cdots + B_r \cos r\omega t$$

The value of A_0, A_r and B_r can be mathematically determined as follows:

For A :→ Multiplying eq^n ① by dt and integrating between the limits 0 & T we have

$$\int_0^T y\,dt = \int_0^T A_0\,dt + A_1\int_0^T \sin\omega t\,dt + \cdots A_r\int_0^T \sin r\omega t\,dt + B_1\int_0^T \cos\omega t\,dt + \cdots + B_r\int_0^T \cos r\omega t\,dt$$

$$= A_0 T + A_1\left[-\frac{\cos\omega t}{\omega}\right]_0^T + \cdots + A_r\left[-\frac{\cos r\omega t}{r\omega}\right]_0^T + B_1\left[\frac{\sin\omega t}{\omega}\right]_0^T + \cdots B_r\left[\frac{\sin r\omega t}{r\omega}\right]_0^T$$

$$A_0 T + \frac{A_1}{\omega}\left[1 - \cos\omega T\right] + \cdots \frac{A_r}{r\omega}\left[1 - \cos r\omega T\right] + \frac{B_1}{\omega}\left[\sin\omega T\right] + \cdots \frac{B_r}{r\omega}\left[\sin r\omega T\right]$$

$$= A_0 T + \frac{A_1}{\omega}\left[1 - \cos 2\pi\right] + \cdots \frac{A_r}{r\omega}\left[1 - \cos 2\pi r\right] + \frac{B_1}{\omega}\left[\sin 2\pi\right] + \cdots \frac{B_r}{r\omega}\left[\sin 2\pi r\right]$$

$$= A_0 T + \frac{A_1}{\omega}\cdot 0 + \cdots \frac{A_r}{r\omega}\cdot 0 + 0 + 0 = A_0 T$$

$$\therefore \boxed{A_0 = \frac{1}{T}\int_0^T y\,dt}$$ This is amplitude of fundamental wave

(11)

Determination of A_r :→ Multiplying equation (1) on both sides by $\sin r\omega t\,dt$ and then integrating it between the limits 0 & T.

$$\int_0^T y \sin r\omega t\, dT = \int_0^T A_0 \sin r\omega t\, dt + A_1\int_0^T \sin\omega t \sin r\omega t\, dt + \cdots + A_r\int_0^T \sin^2 r\omega t\, dt + B_1\int_0^T \cos\omega t \sin r\omega t\, dt + \cdots + B_r\int_0^T \cos r\omega t \sin r\omega t\, dt$$

$$= 0 + \frac{A_1}{2}\int_0^T \left[-\cos \overline{r+1}\,\omega t + \cos \overline{r-1}\,\omega t\right] dt + \cdots + \frac{A_r}{2}\int_0^T \left[1 - \cos 2r\omega t\right] dt + \frac{B_1}{2}\int_0^T \left[\sin \overline{r+1}\,\omega t + \sin \overline{r-1}\,\omega t\right] + \cdots + \frac{B_r}{2}\int_0^T \sin 2r\omega t\, dt$$

$$= \frac{A_1}{2}\left[-\frac{\sin \overline{r+1}\,\omega t}{r+1} + \frac{\sin \overline{r-1}\,\omega t}{r-1}\right]_0^T + \cdots + \frac{A_r}{2}\left[t - \frac{\sin 2r\omega t}{2r\omega}\right]_0^T + \frac{B_1}{2}\left[-\frac{\cos \overline{r+1}}{r+1}\omega t - \frac{\sin \overline{r-1}}{r-1}\omega t\right]_0^T + \cdots + \frac{B_r}{2}\left[-\frac{\cos 2r\omega t}{2r\omega}\right]_0^T$$

$$= \frac{A_r T}{2}$$

$$\therefore \boxed{A_r = \frac{2}{T}\int_0^T y \sin r\omega t\, dt} \quad \text{——— (iii)}$$

This eqn represent the amplitude of cosine terms

Determination of B_r :→ Now Multiplying the eqn (1) by $\cos r\omega t\,dt$ and integrating between the limits, 0 & T we have

$$\int_0^T y \cos r\omega t\, dt = \int_0^T A_0 \cos r\omega t\, dt + A_1\int_0^T \sin\omega t \cdot \cos r\omega t\, dt + \cdots + A_r\int_0^T \sin r\omega t \cos r\omega t\, dt + B_1\int_0^T \cos\omega t \cos r\omega t\, dt + \cdots + B_r\int_0^T \cos^2 r\omega t\, dt$$

$$= 0 + 0 + 0 + \frac{B_r}{2}\left[t - \frac{\sin 2r\omega t}{2r\omega}\right]_0^T$$

$$= \frac{B_r}{2} \cdot T$$

$$\boxed{B_r = \frac{2}{T}\int_0^T y \cos r\omega t \, dt} \quad \text{——— (iv)}$$

Hence Fourier's theorem is mathematically expended as

$$y = f(\omega t) = \frac{1}{T}\int_0^T y \, dt + \frac{2}{T}\int_0^T y \sin^2 \omega t \, dt + \text{----} +$$
$$\frac{2}{T}\int_0^T y \cos^2 \omega t \, dt + \text{----}$$

Application :→

(i) Application to square or rectangular wave :→

A square or rectangular wave is shown in the fig. given which represents a complex Periodic vibration. The displacement of the curve can be split up into two parts viz. for $t = T/2$ and $t = T$ for the first part of the displacement.

$$y = f(t) = C$$

and for the other part for $t = T/2$ to T

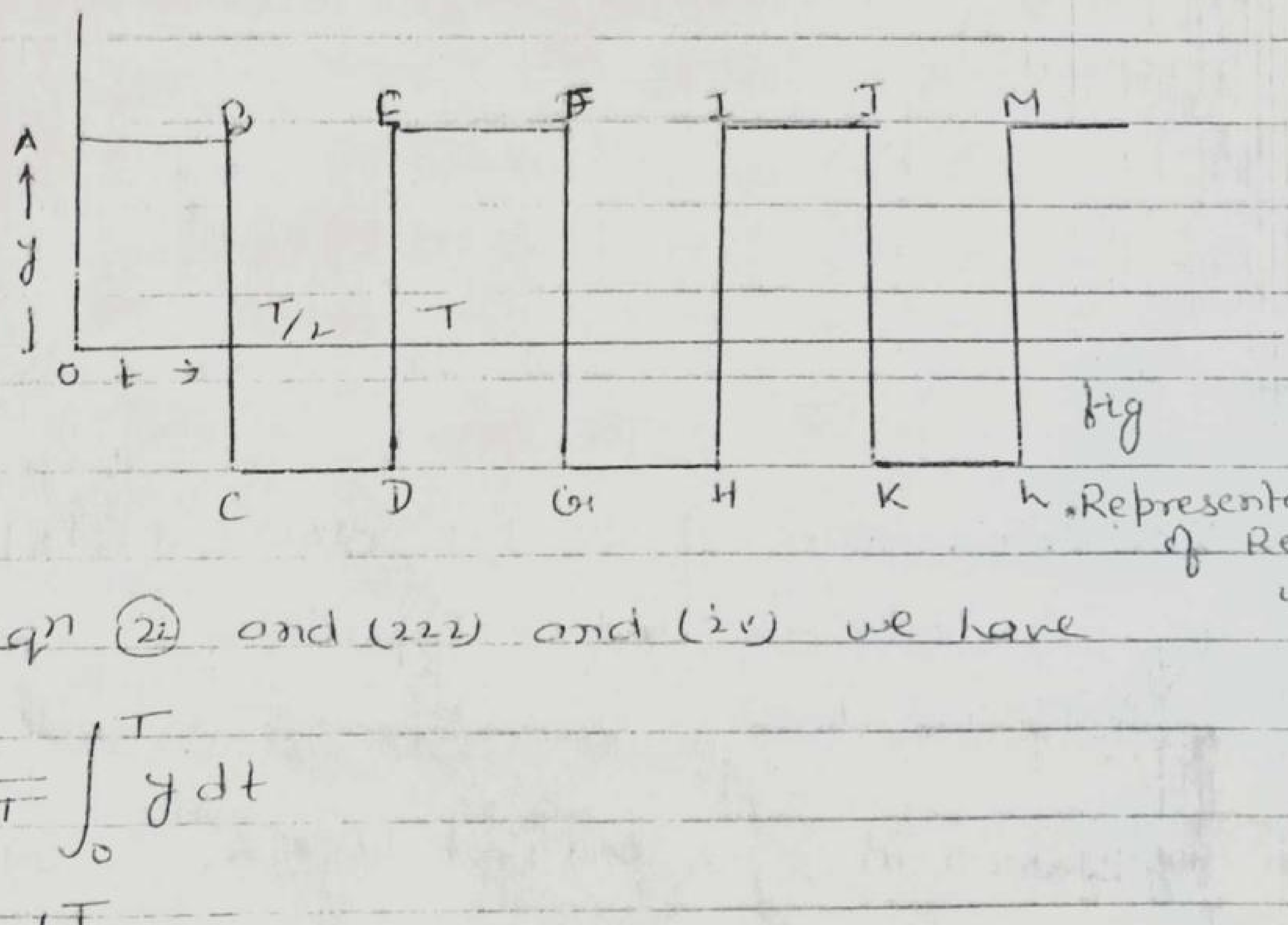

fig. Representation of Rectangular wave

From the eq^n (ii) and (iii) and (iv) we have

$$A_0 = \frac{1}{T}\int_0^T y \, dt$$

$$A_r = \frac{2}{T}\int_0^T y \sin r\omega t \, dt$$

$$B_r = \frac{2}{T}\int_0^T y \cos r\omega t \, dt$$

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Now,

$$y = f(t) = c \quad \text{for } t = 0 \text{ to } t = T/2$$

$$y = f(t) = 0 \quad \text{for } t = T/2 \text{ to } t = T$$

$$\therefore A_0 = \frac{1}{T}\int_0^{T/2} c\,dt = \frac{c}{2}$$

$$A_r = \frac{2}{T}\int_0^{T} c \sin r\omega t\,dt = \frac{2c}{T}\left[-\frac{\cos r\omega t}{r\omega}\right]_0^{T/2}$$

$$= \frac{2c}{r\omega T}(1 - \cos r\pi)$$

If r = odd multiple of π then $\cos r\pi = -1$

If r = even multiple of π then $\cos r\pi = 1$

Therefore cases (i) when r is odd, then

$$A_r = \frac{4c}{T r\omega} = \frac{4c}{2\pi r}$$

(ii) when r is even $A_r = 0$

$$B_r = \frac{2}{T}\int_0^{T/2} y \cos r\omega t\,dt = \frac{2}{T}\int_0^{T/2} c \cos r\omega t\,dt$$

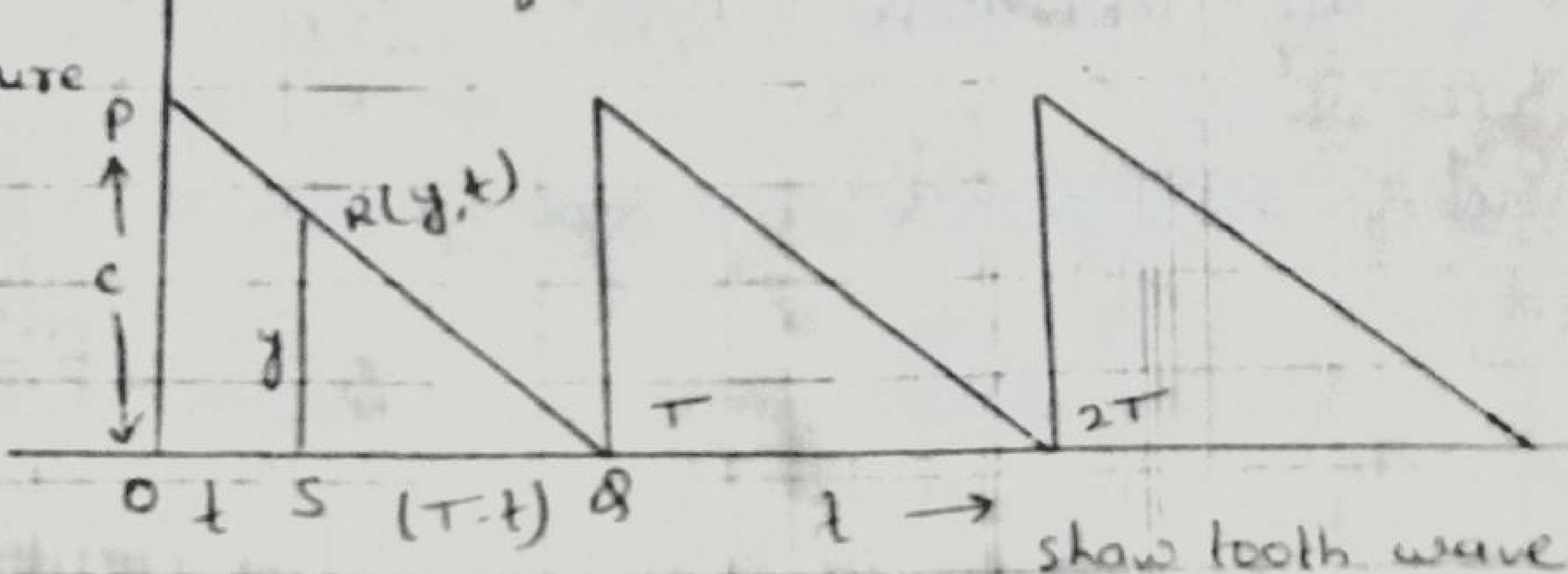
$$= \frac{2c}{T}\left[\frac{\sin r\omega t}{r\omega}\right]_0^{T/2} = \frac{2c}{r\omega T}(\sin \pi r) = 0$$

It means that cosine terms vanish in square wave Hence the complete serie is

$$y = c/2 + \frac{2c}{\pi}\sin \omega t + \frac{2c}{3\pi}\sin 3\omega t + \text{---}$$

Application to show tooth wave :→

A. vibration curve of saw tooth wave is shown in the figure



show tooth wave

Let in the given fig. $OP = c$, $y = c$ at $t = 0$ and $y = 0$ at $t = T, 2T, ---$

Take any point R on the line PQ and let its co-ordinates be (y, t). In the ΔPQO & ΔRQS we have

$$\frac{OP}{RS} = \frac{OQ}{SQ} \quad \text{or,} \quad \frac{c}{y} = \frac{T}{T-t}$$

i.e. $$y = \frac{c(T-t)}{T} = c\left(1 - \frac{t}{T}\right)$$

Now the amplitude $A_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\int_0^T c\left(1 - \frac{t}{T}\right)dt$

or, $$A_0 = \frac{c}{T}\left(T - \frac{T^2}{2T}\right) = c/2$$

$$A_r = \frac{2}{T}\int_0^T c\left(1 - \frac{t}{T}\right)\sin r\omega t\,dt$$

$$= \frac{2c}{T}\int_0^T \left(1 - \frac{t}{T}\right)\sin r\omega t\,dt$$

$$= \frac{2c}{T}\int_0^T \sin r\omega t\,dt - \frac{2c}{T^2}\int_0^T t\sin r\omega t\,dt = \frac{2c}{T^2}\left[-\int_0^T t\sin r\omega t\,dt\right]$$

or, $$A_r = \frac{2c}{T^2}\left\{\left[t\,\frac{\cos r\omega t}{r\omega}\right]_0^T - \int_0^T \frac{\cos r\omega t}{r\omega}\,dt\right\}$$

$$= \frac{2c}{T^2}\left[\frac{T}{r\omega}\cos 2\pi r - \left\{\frac{\sin r\omega t}{r^2\omega^2}\right\}_0^T\right] = \frac{c}{\pi r}$$

And $$B_r = \frac{2}{T}\int_0^T c\left(1 - \frac{t}{T}\right)\cos r\omega t\,dt$$

$$= \frac{2}{T}\left[\int_0^T c\cos r\omega t\,dt - \int_0^T \frac{ct}{T}\cos r\omega t\,dt\right] = \frac{2c}{T^2}\left[-\int_0^T t\cos r\omega t\,dt\right]$$

$$= \frac{2c}{T^2}\left[-t\,\frac{\sin r\omega t}{r\omega}\right]_0^T = -\frac{2c}{T^2}\,\frac{T}{r\omega}\,\sin 2\pi r = 0$$

$\therefore$ cosine term in saw tooth wave vanishes and hence the complete series is

$$y = \frac{c}{2} + \frac{c}{\pi}\sin\omega t + \frac{c}{2\pi}\sin 2\omega t + - --- + - ---$$

—— x ——

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